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Steiner pentagon covering designs ☆

R.J.R. Abel[a], F.E. Bennett[b,*], H. Zhang[c], L. Zhu[d]

[a] *School of Mathematics, University of New South Wales, Kensington, NSW 2033, Australia*
[b] *Department of Mathematics, Mount Saint Vincent University, 166 Bedford Highway, Halifax, Nova Scotia, Canada B3M 2J6*
[c] *Computer Science Department, The University of Iowa, Iowa City, IA 52242, USA*
[d] *Department of Mathematics, Suzhou University, Suzhou 215006, People's Republic of China*

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Abstract

Let K_n denote the complete undirected graph on n vertices. A Steiner pentagon covering design (SPCD) of order n is a pair $(K_n, \mathscr{B})$, where $\mathscr{B}$ is a collection of $c(n)=\lceil n/5\lceil n-1/2\rceil\rceil$ pentagons from K_n such that any two vertices are joined by a path of length 1 in at least one pentagon of $\mathscr{B}$, and also by a path of length 2 in at least one pentagon of $\mathscr{B}$. The existence of SPCDs is investigated. The main approach is to use certain types of holey Steiner pentagon systems. For n even, the existence of SPCDs is established with a few possible exceptions. For n odd, new SPCDs are found which improve an earlier known result. In addition, new results are also found for Steiner pentagon packing designs. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Let K_n be the complete undirected graph on n vertices. A *pentagon system* (PS) of order n is a pair $(K_n, \mathscr{B})$, where $\mathscr{B}$ is a collection of edge disjoint pentagons which partition the edges of K_n. A *Steiner pentagon system* (SPS) of order n is a pentagon system $(K_n, \mathscr{B})$ with the additional property that every pair of vertices are joined by a path of length 2 in exactly one pentagon of $\mathscr{B}$. It is known [10] that the spectrum of SPSs is precisely the set of all $n \equiv 1$ or $5 \pmod{10}$, except $n=15$ for which no such

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* Corresponding author.
E-mail address: frank.bennett@msvu.ca (F.E. Bennett).

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system exists. For other values of n, we have Steiner pentagon covering and packing designs.

A *Steiner pentagon covering* (SPC) of order n is a pair $(K_n, \mathscr{B})$, where $\mathscr{B}$ is a collection of pentagons from K_n such that any two vertices are joined by a path of length 1 in at least one pentagon of $\mathscr{B}$, and also by a path of length 2 in at least one pentagon of $\mathscr{B}$. It is well known that any SPS of order n gives a BIBD on n points with block size $k=5$ and index $\lambda=2$. Similarly, an SPC of order n may lead to a usual covering on n points with $k=5$ and index $\lambda=2$. It is known (see [11]) that such a covering contains at least

$$c(n) = \left\lceil \frac{n}{5} \left\lceil \frac{n-1}{2} \right\rceil \right\rceil$$

blocks. If an SPC(n) contains the minimum number of $c(n)$ pentagons, we call it a *Steiner pentagon covering design* (SPCD), denoted by SPCD(n).

Example 1.1. There exists an SPCD(10) based on the set $\{0,1,\ldots,9\}$, which contains $c(10)=10$ pentagons:

$$\begin{array}{ccccc} (0,2,1,4,5), & (2,4,3,6,7), & (4,6,5,8,9), & (6,8,7,0,1), & (8,0,9,2,3), \\ (0,4,7,9,5), & (2,6,9,1,7), & (4,8,1,3,9), & (6,0,3,5,1), & (8,2,5,7,3). \end{array}$$

It is clear that an SPS(n) is also an SPCD(n). SPS(n)s are known to exist for $n \equiv 1,5 \pmod{10}$ and $n \neq 15$.

Theorem 1.2. *For any positive integer* $n \equiv 1$ *or* $5 \pmod{10}$, *there exists an* SPCD(n), *except for* $n=15$.

The following results are also known [5]:

Theorem 1.3. *For any positive integer* $n \equiv 7 \pmod{10}$, *there exists an* SPCD(n) *except possibly when* $n = 17, 27, 37, 47, 67$ *or* 77.

Theorem 1.4. *For any positive integer* $n \equiv 9 \pmod{10}$, *there exists an* SPCD(n) *except possibly when* $n = 9, 19, 29, 49, 69, 79, 89, 99, 109, 119, 129, 139, 149, 159, 169,$ *or* 189.

The main purpose of this paper is to investigate the existence of SPCDs. Since there are above known results for odd orders, we shall focus on the existence of SPCDs when n is even. The following result is established in this paper:

Theorem 1.5. *For any even integer* $n \geqslant 6$, *there exists an* SPCD(n) *except possibly for* $n \in D$, *where* D *contains the integers in the following table*:

$n \,(\mathrm{mod}\ 10)$	n
0	*None*
2	22, 42, 82
4	14, 74
6	*None*
8	18, 28, 38

For odd n, we give some improvements of Theorems 1.3 and 1.4. In [5], the existence of Steiner pentagon packing designs is established with a few undecided cases. We will improve this by constructing eight new Steiner pentagon packing designs.

2. Auxiliary designs

Our main approach is similar to that in [5]. That is, based on the existence of certain types of HSPSs, we recursively construct large orders of SPCDs using the small orders of SPCDs obtained by computer search.

Let S be a set and $\mathscr{H} = \{S_1, S_2, \ldots, S_n\}$ be a set of subsets of S. A *holey Steiner pentagon system* having *hole set* $\mathscr{H}$ is a triple $(S, \mathscr{H}, \mathscr{B})$ where $\mathscr{B}$ is a collection of pentagons, satisfying the following properties:

1. Two vertices from the same hole S_i do not occur together in any pentagon of $\mathscr{B}$.
2. Two vertices from different holes S_i and S_j $(i \neq j)$ are joined by a path of length 1 in exactly one pentagon of $\mathscr{B}$, and also by a path of length 2 in exactly one pentagon of $\mathscr{B}$.

The *order* of the system is $|S|$.

If $\mathscr{H} = \{S_1, S_2, \ldots, S_n\}$ is a partition of S, then we simply denote the system by HSPS(T), where T is the *type* and defined to be the multiset $\{|S_i| : 1 \leqslant i \leqslant n\}$. We shall use an 'exponential' notation to describe types: so type $t_1^{u_1} \ldots t_k^{u_k}$ denotes u_i occurrences of t_i, $1 \leqslant i \leqslant k$, in the multiset. It is clear that an SPS of order v (or SPS(v)) is equivalent to an HSPS(1^v).

If $\mathscr{H} = \{H\}$, the system is called *incomplete Steiner pentagon system*, denoted by ISPS($|S|, |H|$). It is obvious that an ISPS(n, k) is equivalent to an HSPS($1^{n-k}k^1$).

HSPSs have been useful in the construction of various combinatorial designs such as perfect Mendelsohn designs with block-size 5 (see [4,6]), authentication perpendicular arrays (see [9]), and Steiner pentagon packing designs (see [5]). The existence of HSPSs of uniform type h^n has been investigated by Abel et al. in [1,5]. The following known results are useful in this paper; these results are given in [5].

Theorem 2.1. *Necessary conditions for the existence of an HSPS of type* h^n, *are* (1) $n \geqslant 5$, (2) $n(n-1)h^2 \equiv 0 \,(\mathrm{mod}\ 5)$, *and* (3) *if* h *is odd*, *then* n *must be odd. These* *three conditions are also sufficient, except for* $(h,n)=(1,15)$ *and possibly for the following cases*:

(1) $h=6$ *and* $n\in\{6,36\}$;
(2) $h=15$ *and* $n\in\{19,23,27\}$;
(3) $h=30$ *and* $n\in\{18,22,24\}$;
(4) $h\in\{7,9,11,13,31,37,41\}$ *and* $n=15$.

From an HSPS we can fill in holes to obtain an SPCD as follows, which explains our main approach.

Lemma 2.2. *Suppose there exists an HSPS of type* $(10a)^m(10b)^n(2u)^1$ *and there exists an* SPCD(h) *for* $h=10a,10b,2u$, *where* a,b,u *are nonnegative integers. Then there exists an* SPCD(v) *for* $v=10am+10bn+2u$. *If* $u=2$ *and* $a+b>0$, *then the conclusion holds t*

Proof. The HSPS contains

$$b=\frac{1}{5}\left[\binom{v}{2}-\binom{10a}{2}m-\binom{10b}{2}n-\binom{2u}{2}\right]$$

pentagons. An SPCD$(10a)$ contains $c(10a)=10a^2$ pentagons. An SPCD$(10b)$ contains $c(10b)=10b^2$ pentagons. An SPCD$(2u)$ contains $c(2u)=\lceil 2u^2/5\rceil$ pentagons. It is readily checked that

$$b+mc(10a)+nc(10b)+c(2u)=c(v).$$

Thus, we may construct an SPCD on every hole of the HSPS to obtain the desired SPCD(v). When $u=2$, we do not have an SPCD(4). However, since $a+b>0$, we can adjoin a point to the size 4 hole and construct an SPCD(5) on the five points. Since $c(4)=c(5)=2$, the conclusion still holds. $\square$

The next set of lemmas will be useful when applying the above lemma.

Lemma 2.3 (Bennett et al. [5]). *Suppose there exists a* TD$(6,m)$. *If* $m\equiv 0,1,4\ (\mathrm{mod}\ 5)$, *then there exists an* HSPS *of type* $10^m(2w)^1$ *for* $0<w<m$, *and type* $20^m(2k)^1$ *for* $\eta<k<2m-1$, *where* $\eta=1$ *when* $m\equiv 4\ (\mathrm{mod}\ 5)$ *and* $\eta=0$ *otherwise.*

Lemma 2.4 (Bennett et al. [5]). *For* $t=5$ *or any* $t\geqslant 9$ *and* $t\neq 43,67$, *there always exists an* HSPS(10^tu^1) *or an* HSPS$(20^{t/2}u^1)$ *for* $u=4,6,8$. *For* $t=43,67$, *there exists an* HSPS$(20^{(t-5)/2}10^5u^1)$ *for* $u=4,6,8$.

Lemma 2.5. *Suppose there exists an* HSPS *of type* 1^nu^1 *and there exists an* SPCD(u), *where* u *is odd and* n *is even. Then there exists an* SPCD$(n+u)$.

Proof. The HSPS contains

$$b = \frac{1}{5}\left[\binom{n}{2} + nu\right]$$

pentagons. An SPCD(u) contains $c(u) = \lceil u(u-1)/10 \rceil$ pentagons. It is easy to see that

$$b + c(u) = c(n+u).$$

Thus, we may construct an SPCD(u) on the size u hole of the HSPS to obtain the desired SPCD($n+u$). □

The following HSPSs are useful in this paper.

Lemma 2.6. *For any* $t \geqslant 9$ *and* $t \notin \{10, 12, 14, 18, 22, 23, 27, 28, 32, 38, 42, 43, 47, 48, 52, 62, 67, 72\}$, *there exists an* HSPS *of type* $10^t 12^1$.

Proof. The proof is a simple extension of the proofs of [5, Lemmas 6.1, 6.3–6.5] to cover the case $u = 12$. In most cases, Lemma 2.3 can be applied. □

Lemma 2.7. *Let* $t \equiv 0, 1,$ *or* $4 \pmod 5$. *If* $t > 16$, *then there exists an* HSPS *of type* $10^t 32^1$.

Proof. Apply Lemma 2.3 with $m > 16$ and $w = 16$. This is essentially an extension of [5, Lemma 6.1] to cover the case $u = 32$, where $t > 16$. □

Lemma 2.8. *There exists an* HSPS *of type* $20^5 12^1$.

Proof. Apply Lemma 2.3 with $m = 5$ and $k = 6$. □

If n and k are two integers which are both even or both odd, then in a manner similar to the definition of ISPS(n, k), we have the concept of an incomplete Steiner pentagon covering design (ISPCD). Let Y be a k-subset of an n-set X (of points). K_n and K_k are the complete undirected graphs based on X and Y respectively. An *incomplete Steiner pentagon covering design* is a triple $(X, Y, \mathscr{B})$, where $\mathscr{B}$ is a collection of $c(n) - c(k)$ pentagons from $K_n - K_k$ satisfying the following properties:

1. Two vertices from Y do not occur together in any pentagon of $\mathscr{B}$.
2. Two vertices from X, not both in Y, are joined by a path of length 1 in at least one pentagon of $\mathscr{B}$, and also by a path of length 2 in at least one pentagon of $\mathscr{B}$.

The ISPCD is also denoted by ISPCD(n, k). Y is called a *hole*. Note that an ISPCD($n, 0$) is the same as an SPCD(n).

Example 2.9. There exists an ISPCD$(8,2)$ based on the set $\{1,2,\dots,6\}\cup\{x,y\}$, which contains $c(8)-c(2)=6$ pentagons:

$$(1,2,3,x,4),\quad (1,6,5,2,x),\quad (3,5,x,6,4),$$
$$(1,2,4,y,3),\quad (1,y,2,6,5),\quad (3,4,5,y,6).$$

The purpose of the concept is to construct an SPCD(n). The following lemma is evident.

Lemma 2.10. *Suppose there exists an* ISPCD(n,k). *If* $k=4$ *or there exists an* SPCD(k), *then there exists an* SPCD(n).

Proof. We construct an SPCD(k) $(K_k,\mathscr{A})$ on Y. When $k=4$, we construct an SPCD(5) $(K_5,\mathscr{A})$ on $Y\cup\{x\}$, where $x\in X-Y$. Then, $(K_n,\mathscr{B}\cup\mathscr{A})$ is the desired SPCD(n), where $\mathscr{B}$ is the collection of pentagons in the ISPCD(n,k). $\square$

3. Direct constructions

For our direct constructions, we shall adopt the standard approach of using finite abelian groups to generate the set of blocks (pentagons) for a given design. That is, instead of listing all of the blocks of the design, we shall for the most part list a set of base (or starter) blocks and generate others by an additive group and perhaps some further automorphisms. We shall often use '$+k$' method which means that by adding multiples of $k \pmod{mk}$ we can obtain m pentagons from each starter pentagon. In the case of $n=10$ of the next lemma, the 10 pentagons obtained from the two starter pentagons by using '+2' method are shown in Example 1.1.

Lemma 3.1. *There exists an* SPCD(n) *for* $n=6,8,10,12,20,30,40$.

Proof. SPCD(6) and SPCD(8) are given in [5]. An SPCD(n), based on the set Z_n has the following $c(n)$ pentagons:

$n=10$: $(0,2,1,4,5),\quad (0,4,7,9,5)\quad +2 \bmod 10.$

$n=12$: $(0,3,1,7,11),\quad (3,11,1,2,6),\quad (0,4,5,2,9),$
$(3,1,9,4,10),\quad (3,2,0,6,8)\quad +4 \bmod 12.$

$n=20$: $(0,10,4,13,12),\quad (3,11,1,10,16),\quad (0,11,14,6,2),$
$(3,17,12,9,2),\quad (0,4,2,7,13),\quad (1,9,11,2,5),$
$(2,17,18,16,15),\quad (3,0,15,17,19)\quad +4 \bmod 20.$

$n=30$: $(0,2,19,10,15),\quad (0,6,21,9,1),\quad (0,25,15,1,3),$
$(0,20,2,6,19),\quad (0,11,12,4,27),\quad (0,7,11,5,14)\quad +2 \bmod 30.$

$n=40$: $(0,6,23,8,31)$, $(0,4,16,11,26)$, $(0,10,26,6,3)$,
$(0,27,20,28,33)$, $(0,38,9,29,13)$, $(0,29,1,5,19)$,
$(0,39,37,27,18)$, $(0,21,3,35,1)$ $+2 \bmod 40$. □

Lemma 3.2. *There exists an* ISPCD(n,k) *for* $(n,k)=(16,4),(18,2),(24,4),(26,4),(34,6),(44,6),(48,12),(52,8),(78,12)$.

Proof. These ISPCDs are based on the set $Z_{n-k}\cup\{x_1,x_2,\ldots,x_k\}$. As before, we use the $+2$ or $+4$ method, which means we add only multiples of 2 or $4 \bmod n-k$ to the given base blocks.

$(16,4)$: $(x_1,0,2,4,1)$, $(x_1,3,7,1,2)$, $(x_2,0,3,2,6)$,
$(x_2,3,5,4,9)$, $(x_3,0,4,10,7)$, $(x_3,1,9,11,6)$,
$(x_4,2,5,10,0)$, $(x_4,3,4,11,1)$ $+4 \bmod 12$.

The x_i's act as infinities, that is, $x_i+j=x_i$ for any $1\leqslant i\leqslant 4$ and $0\leqslant j<12$.

$(18,2)$: $(0,1,5,13,15)$, $(0,4,2,8,11)$, $(x_1,0,7,10,15)$,
$(x_2,0,8,1,11)$ $+2 \bmod 16$.

$(24,4)$: $(2,6,1,3,11)$, $(0,3,7,8,1)$, $(2,4,14,8,5)$,
$(x_1,0,16,11,5)$, $(x_1,2,13,6,11)$, $(x_2,0,11,10,3)$,
$(x_2,2,0,5,17)$, $(x_3,0,9,19,2)$, $(x_3,1,2,8,15)$,
$(x_4,0,8,6,18)$, $(x_4,1,15,17,19)$ $+4 \bmod 20$.
$(1,5,9,13,17)$

Here the last block generates no blocks other than itself.

$(26,4)$: $(0,11,1,15,9)$, $(0,12,16,2,15)$, $(x_1,0,5,8,11)$,
$(x_2,0,2,9,7)$, $(x_3,0,16,17,13)$, $(x_4,0,17,18,7)$ $+2 \bmod 22$.

$(34,6)$: $(0,1,19,5,11)$, $(0,4,18,26,15)$, $(x_1,0,21,12,9)$,
$(x_2,0,12,3,7)$, $(x_3,1,24,3,23)$, $(x_3,2,1,6,16)$,
$(x_5,0,6,8,21)$, $(x_5,3,15,17,14)$ $+2 \bmod 28$.

In the above design, we replace x_3,x_5 by x_4,x_6 when adding $2,6,10,\ldots,26$ to the last four blocks:

$(44,6)$: $(0,12,2,18,7)$, $(1,23,5,11,2)$, $(0,3,1,11,19)$,
$(0,2,8,12,25)$, $(x_1,0,5,4,7)$, $(x_2,0,21,24,15)$,
$(x_3,0,8,31,27)$, $(x_4,0,14,25,13)$, $(x_5,0,33,16,9)$,
$(x_6,0,20,35,11)$ $+2 \bmod 38$.

$(48,12)$: $(x_1,0,24,2,18)$, $(x_1,1,5,3,11)$, $(x_3,0,27,17,5)$,
$(x_3,3,34,16,10)$, $(x_5,0,1,7,21)$, $(x_5,3,4,2,30)$,
$(x_7,0,3,14,11)$, $(x_8,0,9,28,13)$, $(x_9,0,15,32,27)$,
$(x_{10},0,26,33,15)$, $(x_{11},0,13,20,31)$, $(x_{12},0,32,19,35)$ $+2 \bmod 36$.

In this design, replace x_1, x_3, x_5 by x_2, x_4, x_6 when adding $2, 6, 10, \ldots, 34$ to the first six blocks:

$$\begin{array}{ll} (52,8): & (0,22,2,14,5),\quad (0,17,39,9,7),\quad (0,28,10,12,9),\\ & (0,1,17,23,3),\quad (x_1,0,11,40,35),\quad (x_2,0,8,41,1),\\ & (x_3,0,29,6,33),\quad (x_4,0,34,11,19)\quad (x_5,0,6,20,13),\\ & (x_5,2,33,7,17),\quad (x_7,0,40,21,22),\quad (x_7,1,33,14,27)\quad +2 \bmod 44. \end{array}$$

In this design, replace x_5, x_7 by x_6, x_8 when adding $2, 6, 10, \ldots, 42$ to the last four blocks:

$$\begin{array}{ll} (78,12): & (0,22,1,34,17),\quad (0,3,17,7,57),\quad (0,25,13,11,34),\\ & (0,28,26,44,36),\quad (3,49,9,39,11),\quad (0,26,36,24,63),\\ & (x_1,0,1,38,33),\quad (x_2,0,24,5,53),\quad (x_3,0,11,18,55),\\ & (x_4,0,52,41,35),\quad (x_5,0,21,46,3),\quad (x_6,0,4,9,51),\\ & (x_7,0,13,6,59)\quad (x_8,0,65,14,23),\quad (x_9,0,31,4,39),\\ & (x_{10},0,50,65,31),\quad (x_{11},0,6,25,21),\quad (x_{12},0,20,19,63)\quad +2 \bmod 66. \end{array}$$

□

Lemma 3.3. *There exists an* SPCD(n) *for* $n = 16, 24, 26, 34, 44, 48, 52, 78$.

Proof. Apply Lemma 2.10 with the ISPCDs provided in Lemma 3.2. Also SPCD(6), SPCD(8) and SPCD(12) are known. □

The following HSPSs will be used later.

Lemma 3.4. *There exists an* HSPS(T) *for* $T = 1^{42}7^1, 2^{18}8^1, 2^{22}4^1, 4^{10}2^1, 4^{10}6^1, 4^{10}8^1, 4^{15}2^1, 4^{15}6^1, 6^{10}4^1, 4^{15}16^1, 4^{18}6^1, 4^{18}16^1$.

Proof. Each of these HSPSs has type $h^n u^1$ for certain values of h, n, u and is based on the point set $Z_{hn} \cup \{x_1, x_2, \ldots, x_u\}$. As before, apply the $+2$ or $+4$ method $\bmod\, hn$ to the given base blocks.

$$\begin{array}{ll} 1^{42}7^1: & (1,7,5,13,33),\quad (0,2,6,16,1),\quad (0,14,30,10,15),\\ & (0,7,31,20,29),\quad (x_1,0,21,2,39),\quad (x_2,0,8,25,13),\\ & (x_3,0,31,18,21),\quad (x_4,0,18,15,11),\quad (x_5,0,25,34,33),\\ & (x_6,0,30,23,7),\quad (x_7,0,36,17,3)\quad +2 \bmod 42. \end{array}$$

$$\begin{array}{ll} 2^{18}8^1: & (0,29,5,21,25),\quad (0,4,26,34,3),\quad (x_1,0,21,8,7),\\ & (x_2,0,16,3,33),\quad (x_3,0,33,24,31),\quad (x_4,1,11,10,22),\\ & (x_5,0,30,4,23),\quad (x_5,1,9,7,26),\quad (x_7,0,31,20,22),\\ & (x_7,3,17,2,29)\quad +2 \bmod 36. \end{array}$$

Here we replace x_5, x_7 by x_6, x_8, respectively, when adding $2, 6, 10, \ldots, 34$, to the last four blocks.

$2^{22}4^1$: $(0,27,13,31,7)$, $(0,4,24,18,3)$, $(0,11,27,39,1)$,
$(0,18,4,16,39)$, $(0,31,6,34,43)$, $(0,15,5,8,13)$,
$(x_1,0,34,9,1)$, $(x_2,0,2,19,17)$, $(x_3,0,33,12,3)$,
$(x_4,0,36,29,25)$ $+2 \bmod 44$.

$4^{10}2^1$: $(0,28,11,6,35)$, $(0,1,38,29,37)$, $(0,11,23,19,25)$,
$(0,14,32,28,9)$, $(0,15,36,9,33)$, $(0,39,1,24,8)$,
$(x_1,0,6,13,27)$, $(x_2,0,38,25,3)$ $+2 \bmod 40$.

$4^{10}6^1$: $(0,2,1,5,19)$, $(0,6,19,12,15)$, $(0,17,11,26,23)$,
$(x_1,0,26,4,39)$, $(x_1,2,35,37,9)$, $(x_3,0,27,18,29)$,
$(x_4,0,28,33,15)$, $(x_5,0,31,2,23)$, $(x_6,0,36,37,21)$ $+2 \bmod 40$.
$(0,8,16,24,32)$, $(0,16,32,8,24)$, $(1,9,17,25,33)$.

Here we add only 0, 2, 4, 6 to the last three blocks. Also, replace x_1 by x_2 when adding $2, 6, 10, \ldots, 38$, to the fourth and fifth blocks.

$4^{10}8^1$: $(0,2,6,28,3)$, $(0,1,3,7,29)$, $(x_1,0,6,5,19)$,
$(x_2,0,13,32,17)$, $(x_3,0,27,18,29)$, $(x_4,0,24,21,33)$,
$(x_5,0,33,38,15)$, $(x_6,0,26,31,25)$, $(x_7,0,19,36,27)$,
$(x_8,0,28,35,11)$ $+2 \bmod 40$.
$(0,8,16,24,32)$.

Here we add only $0, 1, 2, \ldots, 7$ to the last block.

$4^{15}2^1$: $(0,25,21,47,5)$, $(0,33,53,9,19)$, $(0,31,37,49,35)$,
$(0,37,14,27,49)$, $(0,9,18,17,41)$, $(0,12,13,56,7)$,
$(0,43,51,12,32)$, $(0,55,52,26,50)$, $(0,53,32,50,56)$,
$(0,14,22,20,47)$, $(x_1,0,44,41,43)$, $(x_2,0,38,7,39)$ $+2 \bmod 60$.

$4^{15}6^1$: $(0,25,1,34,8)$, $(0,33,23,7,21)$, $(0,19,13,47,5)$,
$(0,1,2,55,42)$, $(0,55,59,39,17)$, $(0,32,54,10,57)$,
$(0,23,46,56,2)$, $(x_1,0,40,11,9)$, $(x_2,0,11,42,31)$,
$(x_3,0,39,32,35)$, $(x_4,0,43,52,33)$, $(x_5,0,56,5,37)$,
$(x_6,0,14,49,57)$ $+2 \bmod 60$.
$(0,12,24,36,48)$, $(0,24,48,12,36)$, $(1,13,25,37,49)$.

Here we add only $0, 2, 4, \ldots, 10$ to the last three blocks:

$6^{10}4^1$: $(0,41,5,16,18)$, $(0,21,27,9,35)$, $(0,6,35,39,7)$,
$(0,11,2,45,53)$, $(0,59,12,37,15)$, $(0,37,42,26,31)$,
$(0,36,4,55,57)$, $(0,14,6,28,13)$, $(x_1,0,39,38,41)$,
$(x_2,0,19,52,15)$, $(x_3,0,56,29,13)$, $(x_4,0,26,43,57)$ $+2 \bmod 60$.
$(0,12,24,36,48)$.

Here we add only $0, 1, 2, \ldots, 11$ to the last block:

$$
\begin{array}{ll}
4^{15}16^1: & (0,11,35,49,32), \quad (x_1,0,22,5,38), \quad (x_1,1,43,44,47),\\
& (x_3,0,9,11,7), \quad (x_3,6,40,58,49), \quad (x_5,3,37,15,21),\\
& (x_5,6,10,56,4), \quad (x_7,0,44,51,26), \quad (x_8,5,49,10,39),\\
& (x_9,0,19,56,27), \quad (x_{10},0,57,20,25), \quad (x_{11},0,35,46,41),\\
& (x_{12},0,50,37,47), \quad (x_{13},0,1,28,21), \quad (x_{14},0,2,23,31),\\
& (x_{15},0,6,19,59), \quad (x_{16},0,20,1,33) \quad +2 \bmod 60.\\
& (0,12,24,36,48), \quad (0,24,48,12,36), \quad (1,13,25,37,49).
\end{array}
$$

Here we add only $0, 2, 4, \ldots, 10$ to the last three blocks. Also, replace x_1, x_3, x_5 by x_2, x_4, x_6 when adding $2, 6, 10, \ldots, 58$, to the 2nd, 3rd, …, 7th blocks.

$$
\begin{array}{ll}
4^{18}6^1: & (0,46,34,31,7), \quad (0,44,15,13,23), \quad (0,40,5,12,11),\\
& (0,35,2,21,61), \quad (0,38,32,42,67), \quad (0,5,11,25,9),\\
& (0,16,24,10,63), \quad (0,59,37,49,15), \quad (0,42,22,26,39),\\
& (0,51,55,63,21), \quad (x_1,0,55,52,7), \quad (x_2,0,70,27,71),\\
& (x_3,0,45,14,61), \quad (x_4,0,24,65,19), \quad (x_5,0,22,23,3),\\
& (x_6,0,49,64,9) \quad +2 \bmod 72.
\end{array}
$$

$$
\begin{array}{ll}
4^{18}16^1: & (0,38,58,23,32), \quad (0,9,13,65,19), \quad (0,26,24,30,33),\\
& (0,21,33,19,29), \quad (x_1,0,12,62,34), \quad (x_1,3,1,51,45),\\
& (x_3,0,8,53,29), \quad (x_4,0,13,60,71), \quad (x_5,0,1,32,39),\\
& (x_6,0,58,43,1), \quad (x_7,0,24,5,69), \quad (x_8,0,5,8,51),\\
& (x_9,0,23,56,55), \quad (x_{10},0,15,20,37), \quad (x_{11},0,59,70,47),\\
& (x_{12},0,56,49,11), \quad (x_{13},0,55,28,3), \quad (x_{14},0,62,41,25),\\
& (x_{15},0,30,61,17), \quad (x_{16},0,68,31,63) \quad +2 \bmod 72.
\end{array}
$$

Here we replace x_1 by x_2 when adding $2, 6, 10, \ldots, 70$, to the 5th and 6th blocks. □

4. Recursive constructions

In the recursive constructions of group divisible designs (GDDs) and pairwise balanced designs (PBDs), the 'weighting' technique and Wilson's Fundamental Construction (see [13]) are quite often used, where we start with a 'master' GDD and small input designs to obtain a new GDD. Similar techniques will be applied in our constructions of HSPSs, where we either start with an HSPS and use transversal designs (TDs) as input designs or start with a GDD and use some HSPSs as input designs. More specifically, we shall make use of the following two constructions, which are taken from [1]. The second is a more general form of Construction 3.3 in [1]. For more background information about GDDs and PBDs the readers are referred to [7,13].

Construction 4.1. *Suppose that both an* HSPS *of type* $\{h_1, h_2, \ldots, h_n\}$ *and a* $\mathrm{TD}(5, m)$ *exist. Then there exists an* HSPS *of type* $\{mh_1, mh_2, \ldots, mh_n\}$.

Construction 4.2. *Let* $(X,\mathscr{G},\mathscr{B})$ *be a* GDD *with groups* $G_1, G_2, \ldots, G_n$. *Suppose there exists a function* $w : X \to \boldsymbol{Z}^+ \cup \{0\}$ (*a weighting function*), *which has the property that for every block* $B=\{x_1,\ldots,x_k\}\in\mathscr{B}$ *there exists an* HSPS *of type* $(w(x_1),\ldots,w(x_k))$. *Then there exists an* HSPS *of type* $\{\sum_{x\in G_1} w(x),\ldots,\sum_{x\in G_n} w(x)\}$.

In the construction of GDDs or PBDs, the technique of 'filling in holes' plays an important role. The technique for HSPSs is described as follows.

Construction 4.3 (Filling in holes).

(1) *Suppose there exists an* HSPS *of type* $\{s_i : 1\leqslant i\leqslant n\}$. *Let* $a\geqslant 0$ *be an integer. For each* $i, 1\leqslant i\leqslant n-1$, *if there exists an* HSPS *of type* $\{s_{ij} : 1\leqslant j\leqslant k_i\}\cup\{a\}$, *where* $s_i = \sum_{1\leqslant j\leqslant k_i} s_{ij}$, *then there is an* HSPS *of type* $\{s_{ij} : 1\leqslant j\leqslant k_i, 1\leqslant i\leqslant n-1\}\cup\{a+s_n\}$.

(2) *Suppose there exists an* HSPS *of type* $\{s_i : 1\leqslant i\leqslant n\}$. *Suppose there exists also an* HSPS *of type* $\{t_j : 1\leqslant j\leqslant k\}$, *where* $s_n = \sum_{1\leqslant j\leqslant k} t_j$. *Then there is an* HSPS *of type* $\{s_i : 1\leqslant i\leqslant n-1\}\cup\{t_j : 1\leqslant j\leqslant k\}$.

We now list some known results for applying the above constructions.

Lemma 4.4 (Colbourn and Dinitz [8, p. 163]). *There exists a* TD$(5,m)$ *for every positive integer* $m\neq 2,3,6,10$.

Lemma 4.5 (Colbourn and Dinitz [8, p. 163]). *There exists a* TD$(6,m)$ *for every positive integer* $m\geqslant 5$ and $m\neq 6,10,14,18,22$.

Lemma 4.6 (Colbourn and Dinitz [8, p. 113]). *There exists a* TD$(q+1,q)$ *for any prime power* q.

Lemma 4.7 (Colbourn and Dinitz [8, p. 163, Table 3.12]). *There exists a* TD$(5,m)-TD(5,k)$ *for every pair of positive integers* (m,k), $m\geqslant 4k$, *and* $(m,k)\neq(6,1),(10,1)$.

The importance of ISPCDs and incomplete transversal designs is seen in the following lemma, which is an analogue of Lemma 3.5 in [11].

Lemma 4.8. *Suppose there exists an* ISPCD$(m+n+w, n+w)$, *where* m *and* $n+w$ *are even and* $m(m+2n+2w)\equiv 0 \pmod 5$. *Further, suppose there exists an incomplete transversal design* TD$(5,m+n)-$TD$(5,n)$. *Then there exists an* ISPCD$(5m+5n+w, 5n+w)$. *If there is also an* SPCD$(5n+w)$, *then there exists an* SPCD$(5m+5n+w)$.

Proof. Let the groups of the incomplete transversal design T be $G_1,\ldots,G_5$, and let the corresponding holes be the sets $H_1,\ldots,H_5$. Let W be a w-set which is disjoint from the point set of T. Construct an SPS(5) on each block of T. From this we obtain $2[(m+n)^2 - n^2] = 2m^2 + 4mn$ pentagons. For $1 \leqslant i \leqslant 5$, construct on each $G_i \cup W$ an ISPCD$(m+n+w, n+w)$ with hole $H_i \cup W$. This provides an additional $5(c(m+n+w) - c(n+w)) = \frac{1}{2}(m^2 + 2mn + 2mw)$ pentagons since $m(m+2n+2w) \equiv 0 \pmod 5$. We note that $2m^2 + 4mn$ and $\frac{1}{2}(m^2 + 2mn + 2mw)$ have a sum of $c(5m+5n+w) - c(5n+w)$. Thus we obtain an ISPCD$(5m+5n+w, 5n+w)$. Filling in the size $5n+w$ hole with an SPCD$(5n+w)$ gives an SPCD$(5m+5n+w)$. □

5. SPCD(v) for v even

We shall establish in this section the existence of an SPCD(v) for v even.

Lemma 5.1. *There exists an* SPCD(v) *for any positive integer* $v \equiv 0 \pmod{10}$.

Proof. Apply Lemma 2.2 with $a = 1$, $m \geqslant 5$ and $b = u = 0$. The required HSPS and SPCD(10) come from Theorem 2.1 and Example 1.1. This solves the cases $v \geqslant 50$. An SPCD(v) also exists from Lemma 3.1 for $v = 10, 20, 30, 40$. □

We may obtain SPCD from ISPCDs. First, we have the next lemma from [5, Lemma 6.8].

Lemma 5.2. *There exists an* SPCD(v) *for* $v = 6, 8$.

Lemma 5.3. *There exists an* SPCD(v) *for* $v = 64, 66$.

Proof. Start with HSPSs of types 12^5 and $6^{10}4^1$ which come from Theorem 2.1 and Lemma 3.4, respectively. In the first case, add four new points to the HSPS and fill in 4 of the holes with an ISPCD(16,4). For the second, add two new points and fill in the holes of size 6 with an ISPCD(8,2). We obtain an ISPCD(64,4) and an ISPCD(66,6). The conclusion now follows from Lemma 2.10. □

Lemma 5.4. *There exists an* SPCD(v) *for* $v = 84, 86, 88$.

Proof. Truncate a TD(6, 8) so as to form a $\{5,6\}$-GDD of type 8^5u^1 for $u = 1,2,3$. Giving weight two to all points we obtain an HSPS(16^5w^1) for $w = 2,4,6$. Add two new points to the HSPS and fill in the size 16 holes with an ISPCD(18,2). This gives an ISPCD$(82+w, w+2)$. The conclusion then follows from Lemma 2.10. □

Lemma 5.5. *There exists an* SPCD(v) *for any positive integer* $v \equiv 4 \pmod{10}$ *and* $v \neq 14, 74$.

Proof. Apply Lemma 2.2 with $a=1$, $b=0$, or 1, and $u=2$. The HSPS has hole size 10, 20 or 4. The required SPCD(10) and SPCD(20) come from Lemma 3.1. Use the HSPSs provided by Lemma 2.4. We obtain an SPCD(v) for $v=10t+4$, where $t=5$ or $t\geqslant 9$. SPCDs for $v=24,34$ and 44 come from Lemma 3.3. An SPCD(64) comes from Lemma 5.3. An SPCD(84) comes from Lemma 5.4. □

Lemma 5.6. *There exists an* SPCD(v) *for any positive integer* $v\equiv 6\,(\mathrm{mod}\,10)$.

Proof. Apply Lemma 2.2 and use the same construction as in Lemma 5.5, with HSPSs supplied in Lemma 2.4. We obtain most of the desired SPCDs. This leaves $v=16,26,36,46,66,76,86$ to be treated; 16, 26, 66 and 86 are solved in Lemmas 3.3, 5.3 and 5.4. For order 46 we start with an HSPS(9^5) and adjoin one infinite point and fill in the holes with an SPCD(10). This produces the required SPCD(46) with $2\times 81+5\times 10=212$ pentagons. For the remaining two cases we apply Lemma 4.8. The required incomplete transversal designs are all from Lemma 4.7. We give the parameters below where the required ISPCDs and SPCDs are all previously known:

m	n	w	$5m+5n+w$
6	1	1	36
12	3	1	76

□

Lemma 5.7. *For every integer* $v\equiv 2$ or $8\,(\mathrm{mod}\,30)$, $v\geqslant 8$, *and* $v\neq 38,218$, *there exists an* SPCD(v).

Proof. For every $m\equiv 0$ or $1\,(\mathrm{mod}\,5)$ and $m\neq 6,36$, there exists an HSPS(6^m) by Theorem 2.1. To this HSPS adjoin two infinite points and fill in the holes, using an ISPCD(8, 2). This gives an ISPCD($6m+2,8$). Since SPCD(8) exists, SPCD($6m+2$) exists and this proves the lemma. □

Lemma 5.8. *There exists an* SPCD(v) *for any positive integer* $v\equiv 8\,(\mathrm{mod}\,10)$ and $v\neq 18,28,38$.

Proof. Apply Lemma 2.2 and use the construction as in Lemma 5.5, with HSPSs supplied in Lemma 2.4. Most of the cases are covered, except for $v=18,28,38,48,68,78,88$. However, an SPCD(48) and an SPCD(78) are constructed in Lemma 3.3, an SPCD(68) comes from Lemma 5.7, and an SPCD(88) is given in Lemma 5.4. □

Lemma 5.9. *There exists an* SPCD(v) *for any positive integer* $v\geqslant 12$, *where* $v\equiv 2\,(\mathrm{mod}\,10)$ *and* $v\neq 22,42,82$.

Proof. We apply Lemma 2.2 with HSPSs supplied in Lemmas 2.6–2.8. Note that v can be expressed in the form $v=10t+12$ or $v=10t+32$ for some suitable nonnegative integer

t. Most of the orders v are covered except for $v=22, 32, 42, 52, 62, 72, 82, 92, 132, 152, 192$. By Lemma 5.7, we obtain an SPCD(v) for v=32, 62, 92, 152. The case v=52 was solved in Lemma 3.3, while $v=72$ can be handled by Lemma 4.8. Use a TD(5, 14) − TD(5, 2) and adjoin 2 infinite points, using ISPCD(16, 4) as input to obtain an ISPCD(72, 12). Since SPCD(12) exists, this also gives an SPCD(72). For $v=132$ or 192, start with an RTD(6, m) for m=11 or 16; this can be viewed as a $\{6, m\}$-GDD of type 6^m. Truncating one group to 4 points gives a $\{5, 6, m-1, m\}$ − GDD($6^{m-1}4^1$). Now give all points in this GDD weight two to obtain HSPSs of types ($12^{10}8^1$) and ($12^{15}8^1$). Finally, we adjoin four infinite points and fill in the holes using ISPCD(16, 4) and SPCD(12) to obtain SPCD(132) and SPCD(192). This completes the proof of the lemma. □

Proof of Theorem 1.5. Combine Lemmas 5.1, 5.5, 5.6, 5.8 and 5.9. □

6. Improvements of SPCD (v) for v odd

For $v \equiv 3 \pmod{10}$ we do not have any example of an SPCD(v) so far. For $v \equiv 1, 5, 7, 9 \pmod{10}$ we do have known results shown in Theorems 1.2–1.4, which we shall update in this section by providing several new designs.

Lemma 6.1. *An* SPCD(v) *exists for* $v = 17, 47, 49, 67, 77, 79, 89$.

Proof. For $v = 17$, start with an HSPS of type 3^5 with holes H_1, H_2, H_3, H_4, H_5. To this HSPS, adjoin two infinite points, say $\{x, y\}$, and form an SPS(5) based on the set $H_i \cup \{x, y\}$ for $i = 1, 2, 3, 4, 5$. It is easy to verify that this gives the desired SPCD(17). Note that the original HSPS contains 18 pentagons and the resulting SPCD will contain $18 + 10 = 28$ pentagons which cover all the 1-apart and 2-apart pairs.

For $v = 47, 49, 67, 79$ an ISPCD(v, 7) is obtainable from the HSPSs of types $4^{10}6^1$, $1^{42}7^1$, $4^{15}6^1$, $4^{18}6^1$, while for $v = 77, 89$ an ISPCD(v, 17) is obtainable from HSPSs of types $4^{15}16^1$ and $4^{18}16^1$. These HSPSs are all given in Lemma 3.4. Now, apply Lemma 2.10. □

Lemma 6.2. *An* SPCD(v) *exists for* $v = 69, 109, 129, 149, 169, 189$.

Proof. For all of the stated values of v, we now have v in a (v, $\{5, 17^*\}$, 1) PBD. The case $v = 69$ is well known and comes from adjoining 17 infinite points to an RBIBD(52, 4, 1). The other values are more recent results (see [2]). In any event, we can obtain an HSPS($1^{v-17}17^1$) for all the above mentioned values of v and then fill in the hole with an SPCD(17) for the desired result by Lemma 2.5. □

Lemma 6.3. *There does not exist an* SPCD(9).

Proof. An SPCD(v) implies the existence of a bicover of pairs by quintuples on v points. The nonexistence of such a bicover is shown in [12]. The conclusion then follows. □

We can now update the known results of Theorems 1.2–1.4 as follows.

Theorem 6.4. *For any positive odd integer* $n \not\equiv 3 \pmod{10}$, *there exists an* SPCD(n) *except for* $n = 9, 15$ *and possibly for* $n \in F$, *where* F *contains the integers in the following table*:

$n \pmod{10}$	n
1	None
5	None
7	27,37
9	19,29,99,119,139,159

7. Steiner pentagon packing designs update

A *Steiner pentagon packing* (SPP) of order n is a pair $(K_n, \mathscr{B})$, where $\mathscr{B}$ is a collection of pentagons from K_n such that any two vertices are joined by a path of length 1 in at most one pentagon of $\mathscr{B}$, and also by a path of length 2 in at most one pentagon of $\mathscr{B}$. As mentioned earlier, any SPS of order n gives a BIBD on n points with block size $k = 5$ and index $\lambda = 2$. An SPP of order n may lead to the ordinary packing on n points with $k = 5$ and index $\lambda = 2$. It is known (see [14,3]) that such a packing contains at most

$$p(n) = \left\lfloor \frac{n}{5} \left\lfloor \frac{n-1}{2} \right\rfloor \right\rfloor - \varepsilon$$

blocks, where $\varepsilon = 1$ for $n \equiv 7$ or $9 \pmod{10}$ and $\varepsilon = 0$ otherwise. If an SPP(n) contains the maximum number of $p(n)$ pentagons, we call it a *Steiner pentagon packing design* (SPPD), denoted by SPPD(n). The existence of SPPD has been investigated by the authors in [5]. The next lemma provides eight new orders for existence.

Lemma 7.1. *There exists an* SPPD(n) *for* $n = 44, 48, 43, 47, 49, 63, 67, 79$.

Proof. Using the HSPSs of types $2^{18}8^1$, $2^{22}4^1$, $4^{10}2^1$, $4^{10}6^1$, $1^{42}7^1$, $4^{15}2^1$, $4^{15}6^1$, $4^{18}6^1$, in Lemma 3.4, we will obtain, respectively, SPPD(n) for $n = 44, 48, 43, 47, 49, 63, 67, 79$. □

We can now update the known results of [5] in the following.

Theorem 7.2. *There always exists an* SPPD(n) *except for* $n = 9, 13, 15$ *and possibly for* $n \in E$, *where* E *contains the integers in the following table*:

n (mod 10)	n
4	24,34,84
6	16,36
8	18
3	33,73,83
7	17,27,37,77
9	19,29,69,89,99,109,119,129,139,149,159,169,189

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